Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 5, pp. 745–747, 2002

APPROXIMATE FORMULAS FOR SAGGING DEFLECTION OF AN ELASTIC ROD UNDER TRANSVERSE LOADING

Yu. V. Zakharov,[1] K. G. Okhotkin,[2] and A. Yu. Vlasov[3]

UDC 539.3

Sagging deflection of a thin elastic rod under transverse loading is expressed approximately in terms of elementary functions. Series expansions of the complete and incomplete elliptic integrals of the first and second kind in the neighborhood of $k^2 = 1/2$ are obtained.

Introduction. Approximate formulas for deflections of compressed flexible rods were proposed in many works. Astapov [1] (see also [2, 3]) reviewed in detail the existing approximate formulas for sagging deflection of a rod under axial compression, applicable for loads exceeding the critical value by no more than 10%. In [1], formulas that approximate well the deflection–load relation for any magnitude of the load were obtained by expanding elliptic integrals.

In the theory of strength of materials, a linear deflection–load relation [4] is valid for a rod under transverse loading if the deflection is smaller than 1% of the rod length. To the authors' knowledge, no approximate formulas applicable for high values of transverse loads are available in the literature. Zakharov and Zakharenko [5] obtained exact formulas for sagging deflection of a rod under transverse loading, expressed in terms of the elliptic Jacobi functions, which can be used to derive approximate formulas suitable for engineering applications.

Formulation of the Problem. In accordance with [5], the exact expression for the first-mode sagging deflection of the free end of a rod under transverse loading is written in the parametric form as

$$f \equiv y/L = 1 - 4(E(k) - E_1(k))/(\pi\sqrt{\lambda}), \qquad \lambda = (2/\pi)^2(K(k) - F_1(k))^2, \tag{1}$$

where $f < 1$ is the dimensionless deflection, $\lambda \equiv P/P_c$ is the dimensionless load, P is the load applied, $P_c = (\pi/2)^2 EI/L^2$ is the Euler force (EI is the flexural rigidity of the rod and L is the rod length), and $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively ($1/2 \leqslant k^2 \leqslant 1$ is the modulus of elliptic integrals).

For $\varphi = \arcsin(1/(k\sqrt{2}))$, we denote the incomplete elliptic integrals of the first and second kinds by

$$F_1(k) \equiv F\Big(\arcsin\frac{1}{k\sqrt{2}}, k\Big) = \int\limits_0^{1/(k\sqrt{2})} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \quad E_1(k) \equiv E\Big(\arcsin\frac{1}{k\sqrt{2}}, k\Big) = \int\limits_0^{1/(k\sqrt{2})} \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt,$$

respectively. The following algorithm is used:

1) Expand the differences of elliptic integrals of the first and second kinds which enter (1) into series with respect to the small parameter $k^2 - 1/2$;

2) Retaining terms up to the third power in the series, find an approximate relation between the modulus k and dimensionless load λ using the Cardan formula.

3) Substituting the resulting expression into (1), obtain an approximate deflection–load relation.

---

[1]Kirenskii Institute of Physics, Siberian Division, Russian Academy of Sciences, Krasnoyarsk 660036. [2]Siberian State Technology University, Krasnoyarsk 660049. [3]Siberian Aerospace Academy, Krasnoyarsk 660014.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 5, pp. 132–134, September–October, 2002. Original article submitted April 29, 2002.

TABLE 1

Formula number	$\lambda = 0.01$	$\lambda = 0.1$	$\lambda = 0.7489$	$\lambda = 1.1834$	$\lambda = 4$
(1)	0.008879	0.08343	0.4707	0.5977	0.8101
(7)	0.008845	0.08095	0.4604	0.6047	—
(10)	0.008038	0.07904	0.4734	0.6040	0.8002

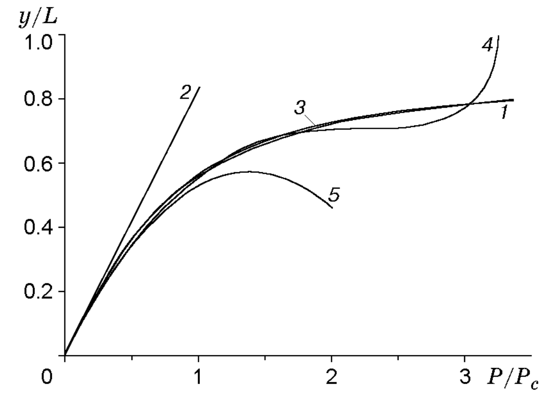

Fig. 1. Sagging deflection versus load: curve 1 refers to the exact relation (1) and curves 2–5 refer to the approximate relations (9), (10), (7), and (8), respectively.

Expansions of the elliptic integrals into series with respect to the small parameter $k-\sqrt{2}/2$ have the form

$$E - E_1 \approx 2^{1/4}\sqrt{k-\sqrt{2}/2}\,\Big[1 - 13\sqrt{2}(k-\sqrt{2}/2)/12 + 17(k-\sqrt{2}/2)^2/240 + \ldots\Big], \tag{2}$$

$$K - F_1 \approx 2\cdot 2^{1/4}\sqrt{k-\sqrt{2}/2}\,\Big[1 + \sqrt{2}(k-\sqrt{2}/2)/4 + 497(k-\sqrt{2}/2)^2/240 + \ldots\Big] \tag{3}$$

and those with respect to the parameter $k^2 - 1/2$ have the form

$$E - E_1 \approx \sqrt{k^2-1/2}\,\Big[1 - 4(k^2-1/2)/3 + 16(k^2-1/2)^2/15 + \ldots\Big], \tag{4}$$

$$K - F_1 \approx 2\sqrt{k^2-1/2}\,\Big[1 + 16(k^2-1/2)^2/15 + 256(k^2-1/2)^4/105 + \ldots\Big]. \tag{5}$$

An analysis shows that expansions (2) and (3) provide higher accuracy in determining sagging deflection compared to expansions (4) and (5).

We retain first two terms in (3). Using the second expression in (1), we obtain the equation

$$2\cdot 2^{1/4}\sqrt{k-\sqrt{2}/2} + 2^{3/4}(k-\sqrt{2}/2)^{3/2}/2 - \pi\sqrt{\lambda}/2 = 0.$$

Solving this cubic equation for $\sqrt{k-\sqrt{2}/2}$ by the Cardan formula and expanding the resulting expression into a series with respect to the parameter λ, we arrive at the approximate relation between the modulus k and the dimensionless modulus λ:

$$k - \sqrt{2}/2 \approx (\pi/2)^2(\sqrt{2}\lambda/8 - \sqrt{2}\lambda^2/64 + \ldots). \tag{6}$$

We substitute the first two terms in expansion (2) for the difference $E - E_1$ into the expression for sagging deflection (1). Then, with allowance for (6), we obtain the approximate dependence of sagging deflection on the load λ:

$$f = 1 - \sqrt{1-\pi^2\lambda/32}[1 - 13(\pi^2\lambda - \pi^4\lambda^2/32)/192]. \tag{7}$$

Expanding this expression into a power series with respect to λ, we simplify formula (7)

$$f = \pi^2\lambda/12 - 25\pi^4\lambda^2/8192. \tag{8}$$

In expansion (8), the linear term corresponds to the following formula known in the theory of strength of materials [4]:

$$f = \pi^2\lambda/12. \tag{9}$$

Using asymptotic approximations, one can obtain an approximate relation whose coefficients are determined by the nonlinear-regression method. Compared to the relations given above, this formula provides higher accuracy in approximating the deflection–load relation for loads exceeding the critical value by more than a factor of three:

$$f = 0.132(\exp{(2 - 2/(1.4\lambda + 1)^2)} - 1). \tag{10}$$

Figure 1 shows the exact and approximate relations for sagging deflection of a cantilever. One can see that formula (9) approximates the dependence $y \sim p$ for deflections of about 20% of the rod length.

The deflections f calculated by formulas (1), (7), and (10) are listed in Table 1. For $\lambda = 4$, the deflection calculated by formula (7) is greater than unity, which is physically meaningless.

The authors are grateful to L. I. Shkutin and V. G. Sukhovol'skii for their attention to the work.

This work was supported by the Russian Foundation for Fundamental Research (Grant Nos. 02-01-01017 and 02-01-06308).

REFERENCES

1. N. S. Astapov, "Approximate formulas for deflections of compressed flexible bars," *J. Appl. Mech. Tech. Phys.*, **37**, No. 4, 573–576 (1996).
2. N. S. Astapov, "An approximation of the form of a compressed flexible rod," *J. Appl. Mech. Tech. Phys.*, **40**, No. 3, 535–538 (1999).
3. A. V. Anfilof'ev, "Mid-span deflection and end shortening of a rod after buckling," *J. Appl. Mech. Tech. Phys.*, **42**, No. 2, 352–357 (2001).
4. G. S. Pisarenko, A. P. Yakovlev, and V. V. Matveev, *Handbook on Strength of Materials* [in Russians], Naukova Dumka, Kiev (1975).
5. Yu. V. Zakharov and A. A. Zakharenko, "Dynamic instability in the nonlinear problem of a cantilever," *Vychisl. Tekh.*, **4**, No. 1, 48–54 (1999).